Dynamic Forces Exerted by Oscillating Cables

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The manner in which the forces applied by a mooring cable to a floating buoy affect (in a linearized sense) the response of that body to regular waves is examined first by considering the case of a vertical, elastic cable of negligible weight density. Seeming infinities are encountered in the cable-alone solutions in view of neglect to the cable-motion-induced hydrodynamic damping forces. Bounded solutions for the buoy-cable system are shown to exist nevertheless. An outline of the treatment of the case of a cable of quite arbitrary scope and combination of parameters is given to demonstrate that the terms contributed by the cable can be identified and studied separately to determine their relative importance. A single set of results for cable force operators is given to show their frequency behavior. Much more numerical experience is needed. Further determinations should be made of cable force operators by measurements of the forces required to oscillate harmonically full-scale cables in a deep river with a known (or measurable) current profile. It is conjectured that, in some cases, cable forces may be approximated by their static force derivatives in the wave-length range of interest. Verification of the validity of this conjecture requires examination of computer results for practical ranges of cable parameters. A new derivation of the equations of motion is given in Appendix A. It is found that the fluid added mass effects as given in other developments are incorrect. In addition, it is shown the effect of buoyancy has not heretofore generally been properly included in both the static and dynamic equations. The influence of these corrections on the dynamic response and the static shape requires further numerical evaluations by employing a suitably modified computer program.

Nomenclature

A = cable cross-sectional area
$\bar{A}$ = complex amplitude of solution, see Eq. (26)
$\bar{B}$ = complex amplitude of solution, see Eq. (27)
c = celerity of transverse waves along cable
D = determinant of coefficients of Eq. (41)
E = Young's modulus of cable material
e = equilibrium length of cable
$\bar{M}_m$ = moment due to mooring or cable forces on buoy
$\bar{M}_w$ = complex amplitude of pitching moment due to wave
s = arc length along relaxed cable to any element
t = time
T = the total tension before Eq. (6); thereafter, the time-dependent part as primes are omitted
T' = the time-dependent part of the cable tension, see Eq. (5); afterwards the primes on all time-dependent parts are dropped for convenience
T_0 = the equilibrium tension
u = cable element velocity along the cable
v = cable element velocity normal to cable
w,w' = cable weight per unit length in vacuo, and in the fluid, respectively
x = complex amplitude of surge response
$\bar{X}_m$ = horizontal force applied by cable to buoy (complex amplitude)
$\bar{X}_w$ = complex amplitude of surge force on buoy due to wave
$\bar{z}$ = complex amplitude of heave force on buoy due to wave
z_c = axial distance between mooring point and buoy center of gravity
$\bar{Z}_m$ = complex amplitude of heave force applied to buoy by cable
$\bar{Z}_w$ = complex amplitude of heave force produced by exciting wave on buoy
α_{ij} = coefficients of $\bar{x}$, $\bar{z}$, and θ in equations of motion of buoy without the cable. Compare Eqs. (38-40) with Eq. (41)
β,γ = cable characteristic functions defined under Eq. (41)
ϵ = strain at any section, ϵ_0 equilibrium value
ϵ' = time-dependent part primes dropped after Eq. (5)
ϕ = angle from horizontal made by an element of cable
ϕ' = time-dependent part [primes dropped after Eq. (5)]
ϕ_0 = equilibrium cable angle
$\bar{\theta}$ = complex amplitude of pitch response
ω = frequency of surface waves
κ = celerity of longitudinal waves in cable
μ = mass of cable per unit of length
μ' = virtual mass of cable per unit length in direction normal to the cable element

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Solutions of the Linearized Equations of a Cable

Motions and Tensions

IF we wish to examine some central features of the coupled motions of a buoy-cable system without the complications involved in developing a complete solution requiring the use of a computer, it is necessary to examine some simple cases which can be solved by "hand-turned mathematics." Numerical solutions for such special cases can then serve as necessary requirements for elaborate computer solutions which are designed to be applicable to more general, as well as simple, cases.

With this objective in mind, let us consider a simple mooring in which the cable has a relatively large static tension in a weak current so that the equilibrium cable geometry is virtually straight and vertical. The bouy is considered to be moored in regular long-crested waves being allowed freedom in surge, heave and pitch. If we ignore the terms involving weight and transverse, as well as longitudinal hydrodynamic, forces on the cable, the equations developed in Appendix A reduce to

$$\mu\frac{\partial u}{\partial t} - \mu\frac{v\partial\phi}{\partial t} - \frac{\partial T}{\partial s} = 0$$

(Summation of tangential forces) (1)

$$\mu'\frac{\partial v}{\partial t} + \mu\frac{u\partial\phi}{\partial t} - \frac{T\partial\phi}{\partial s} = 0$$

(Summation of normal forces) (2)

$$\frac{\partial v}{\partial s} + u\frac{\partial\phi}{\partial s} - (1+\epsilon)\frac{\partial\phi}{\partial t} = 0$$

(Kinematic relation, normal) (3)

$$\frac{\partial u}{\partial s} - v\frac{\partial\phi}{\partial s} - \frac{1}{AE}\frac{\partial T}{\partial t} = 0$$

(Kinematic relation, tangential) (4)

The velocities, tension T, cable angle ϕ, (measured from the horizontal) and strain ϵ are written as the sum of the equilibrium values and the excursions from equilibrium, i.e.,

$$u = u'$$
$$v = v'$$
$$\phi = \phi_0 + \phi' \qquad (5)$$
$$T = T_0 + T'$$
$$\epsilon = \epsilon_0 + \epsilon'$$

Insertion of these into Eqs. (1–4) yields (after dropping the primes for simplicity)

$$\mu\partial u/\partial t - \mu v\partial\phi/\partial t - \partial T_0/\partial s - \partial T/\partial s = 0 \qquad (6)$$

$$\mu'\partial v/\partial t + \mu u\partial\phi/\partial t - (T_0 + T)(\partial\phi_0/\partial s + \partial\phi/\partial s) = 0 \qquad (7)$$

$$\partial u/\partial s - v\partial\phi_0/\partial s - v\partial\phi/\partial s - (1/AE)\partial T/\partial t = 0 \qquad (8)$$

$$\partial v/\partial s + u(\partial\phi_0/\partial s\ \partial\phi/\partial s) - (1 + \epsilon_0 + \epsilon)\partial\phi/\partial t = 0 \qquad (9)$$

If we note that, for a cable which is straight in equilibrium, $\partial\phi_0/\partial s = 0$, and, since weight is neglected, we may take $\partial T_0/\partial s = 0$, then the equations reduce to

$$\mu\partial u/\partial t - \mu v\partial\phi/\partial t - \partial T/\partial s = 0 \qquad (10)$$

$$\mu'\partial v/\partial t + \mu u\partial\phi/\partial t - (T_0 + T)\partial\phi/\partial s = 0 \qquad (11)$$

$$\partial u/\partial s - v\partial\phi/\partial s - (1/AE)\partial T/\partial t = 0 \qquad (12)$$

$$\partial v/\partial s + u\partial\phi/\partial s - (1 + \epsilon_0 + \epsilon)\partial\phi/\partial t = 0 \qquad (13)$$

As is customary, we now neglect all terms involving products of perturbation quantities, such as $v\partial\phi/\partial t$, $u\partial\phi/\partial t$, etc. The equations then reduce to

$$\mu\partial u/\partial t - \partial T/\partial s = 0 \qquad (14)$$

$$\mu'\partial v/\partial t - T_0(\partial\phi/\partial s) = 0 \qquad (15)$$

$$\partial u/\partial s - (1/AE)\partial T/\partial t = 0 \qquad (16)$$

$$\partial v/\partial s - (1 + \epsilon_0)\partial\phi/\partial t = 0 \qquad (17)$$

Solutions of these equations which meet conditions at the anchor, $u(0,t) = 0$ and $v(0,t) = 0$, and specified simple harmonic motions at the upper end, are

$$u = \bar{A}\sin(\omega s/\kappa)e^{i\omega t} \qquad (18)$$

$$v = \bar{B}\sin(\omega s/c)e^{i\omega t} \qquad (19)$$

$$\phi = (i\bar{B}/(1+\epsilon_0)c)\cos(\omega s/c)e^{i\omega t} \qquad (20)$$

$$T = -(iAE/\kappa)\bar{A}\cos\omega s/c e^{i\omega t} \qquad (21)$$

Here $\bar{A}$ and $\bar{B}$ are complex velocity amplitudes which must be determined from the kinematic conditions at the buoy and

$$\kappa = \left(\frac{AE}{\mu}\right)^{1/2} \qquad (22)$$

the celerity of longitudinal elastic waves in the cable

$$c = \left(\frac{T_0}{(1+\epsilon_0)\mu'}\right)^{1/2} \qquad (23)^\dagger$$

the celerity of transverse waves in the cable.

[That Equations (18–21) satisfy (14–17) can be readily verified when using the definitions (22) and (23).]

The kinematic conditions at the buoy are that the velocity components of the end of the cable are given by the resolved velocity components of the buoy. These conditions are, to within the approximations employed here

$$u(l,t) = u_x\cos\phi_0 + u_z\sin\phi_0 = u_z$$

$$v(l,t) = -u_x\sin\phi_0 + u_z\cos\phi_0 = -u_x$$

where u_x and u_z are the velocity components of the buoy at the cable attachment point; then, in terms of the velocities of the center of mass and the pitch angular velocity θ, we have

$$u(l,t) = i\omega(\bar{z} + x_c\bar{\theta})e^{i\omega t} \qquad (24)$$

$$v(l,t) = i\omega(\bar{x} - z_c\bar{\theta})e^{i\omega t} \qquad (25)$$

where x_c, z_c are the coordinates of the coordinates of the attachment point relative to the buoy center of mass (z_c is reckoned positive upward, so if point is below the center of mass a negative value is inserted for z_c).

Using Eqs. (18) and (19) with (24) and (25) yields

$$\bar{A} = i\omega(\bar{z} + x_c\bar{\theta})e^{i\omega t}/\sin(\omega l/\kappa) \qquad (26)$$

$$\bar{B} = -i\omega(\bar{x} - z_c\bar{\theta})e^{i\omega t}/\sin(\omega l/c) \qquad (27)$$

when these values are inserted, Eqs. (18–21) read as follows:

$$u = i\omega(\bar{z} + x_c\bar{\theta})e^{i\omega t}\frac{\sin(\omega s/\kappa)}{\sin(\omega l/\kappa)} \qquad (28)$$

$$v = -i\omega(\bar{x} - z_c\bar{\theta})e^{i\omega t}\frac{\sin(\omega s/c)}{\sin(\omega l/c)} \qquad (29)$$

$$\phi = -\frac{\omega(\bar{x} - z_c\bar{\theta})e^{i\omega t}}{(1+\epsilon_0)c}\frac{\cos(\omega s/c)}{\sin(\omega l/c)} \qquad (30)$$

$$T = \frac{\omega AE(\bar{z} + x_c\bar{\theta})e^{i\omega t}}{\kappa}\frac{\cos(\omega s/\kappa)}{\sin(\omega l/\kappa)} \qquad (31)$$

It might be thought (and expected, perhaps) that this rudimentary theory which ignores all damping mechanisms insofar as the cable is concerned may lead to infinite tensions. It might be surmised from Eq. (31) that $T \rightarrow \infty$ for $\omega l/\kappa \rightarrow n\pi$ ($n = 1,2,3\ldots$). However, it must be observed

†In most cases $\epsilon_0 \ll 1$ and consequently may be dropped.

that the tension depends as well upon the heave amplitude $\bar{z}$ which, in turn, depends upon the two *sets* of eigen frequencies $\omega l/c = n\pi$ and $\omega l/\kappa = n\pi$. Thus, a decision cannot be made regarding the cable tensions, nor other responses, without first solving for the buoy motions. In order to do this, it is necessary to find the horizontal and vertical forces (and the moment of these forces) which the cable applies to the buoy.

Forces and Moment Exerted by Cable on Buoy

The sum of the equilibrium and time-dependent *vertical forces* acting on the buoy at the cable attachment point is

$$(Z_{m_0} + Z_m(t)) = -(T_0 + T)\sin\left(\frac{\pi}{2} + \phi\right) \tag{32}$$

where account is taken of the fact that the forces acting on the buoy are opposite to those acting on the cable. Hence, for ϕ small

$$Z_{m_0} = -T_0$$
$$Z_m(t) = -T(l,t)$$

and, using Eq. (31), the vertical force applied by the cable to the buoy is

$$Z_m(t) = -\mu\omega\kappa(\bar{z} + x_c\bar{\theta})e^{i\omega t}\cot(\omega l/\kappa) \tag{33}$$

The *horizontal force* is a function of the *equilibrium* tension T_0 and the time-dependent angle of the cable $\phi(l, t)$ as may be seen from the following:

$$[(X_{m_0} + X_m(t))] = -[T_0 + T(l,t)]\cos(\phi_0 + \phi)$$

and for $\phi_0 \sim \pi/2$ and ϕ small

$$X_{m_0} = -T_0 \cos\phi_0 \sim 0$$
$$X_m(t) = T_0\phi(l,t) + T(l,t)\phi(l,t) \tag{34}$$

Neglecting the last term (which varies as $e^{2i\omega t}$) in keeping with the previous approximations, the time-dependent horizontal force applied by the cable to the buoy is

$$X_m = -\mu'\omega c(\bar{x} - z_c\bar{\theta})e^{i\omega t}\cot(\omega l/c) \tag{35}$$

by use of Eq. (30).

The time-varying moment acting on the buoy is to first order

$$M_m = x_c Z_m - z_c X_m \qquad M_m = x_c Z_m - z_c X_m \tag{36}$$

Using Eqs. (33) and (35)

$$M_m = \{-\mu\omega\kappa x_c(\bar{z} + x_c\bar{\theta})\cot\left(\frac{\omega l}{\kappa}\right) - \mu'\omega c z_c(\bar{x} - z_c\bar{\theta})\cos\left(\frac{\omega l}{c}\right)e^{i\omega t}\} \tag{37}$$

The Response of the Buoy

The foregoing expressions for the forces produced on the body by the cable are now inserted in the right sides of equations for the motion of the buoy as given by Eq. (63) of Ref. 1. When transposed to the left sides and separated into terms associated with the complex motion amplitudes $\bar{x}$, $\bar{z}$, and $\bar{\theta}$) (i.e., $x = \bar{x}e^{i\omega t}$, etc.), the buoy equations read as follows:

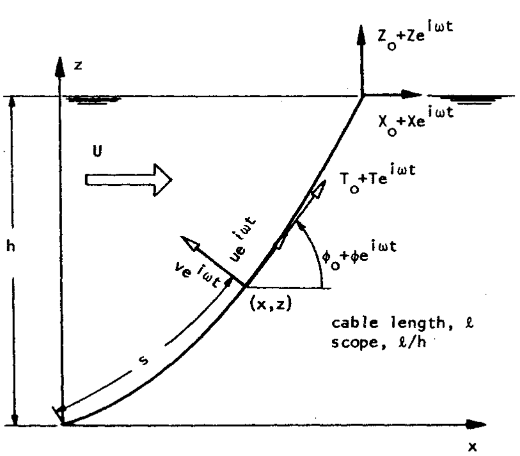

Fig. 1 Definition of coordinates, velocity, and force components.

$$(-\omega^2 A_{11} + i\omega A_{12} + \underline{\omega\mu' c\cot\frac{\omega l}{c}})\bar{x} - \underline{(\omega^2 A_{17} + \omega\mu' c z_c\cot\frac{\omega l}{c})}\bar{\theta} = \bar{X}_w \tag{38}$$

$$(-\omega^2 A_{24} + i\omega A_{25} + A_{26} + \underline{\omega\mu\kappa\cot\frac{\omega l}{\kappa}})\bar{z} + (-\omega^2 A_{27} + i\omega A_{28} + A_{29} + \underline{\mu\omega\kappa x_c\cot(\frac{\omega l}{\kappa})})\bar{\theta} = \bar{Z}_w \tag{39}$$

$$(-\omega^2 A_{31} \underline{- \omega\mu' c z_c\cot\frac{\omega l}{c}})\bar{x} + (-\omega^2 A_{34} + i\omega A_{35} + A_{36} + \underline{\omega\mu\kappa x_c\cot\frac{\omega l}{\kappa}})\bar{z} + (-\omega^2 A_{37} + i\omega A_{38} + A_{39} + \underline{\omega\mu\kappa x_c^2\cot\frac{\omega l}{\kappa}} + \underline{(\omega\mu' c z_c^2\cot\frac{\omega l}{c}})\bar{\theta} = \bar{M}_w \tag{40}$$

The underscored terms are those arising from the cable. It is as though we have a new buoy (without a cable) with modified body coefficients; the character of each is identifiable. For example, the new term $\omega\mu' c\cot(\omega l/c)\bar{x}$ represents a spring or restoring, frequency-dependent force opposing the horizontal motion of the buoy. Similarly, the term $\omega\mu\kappa\cot(\omega l/\kappa)\bar{z}$ is a restoring moment in pitch. The other underscored terms are frequency-dependent cross-coupling coefficients.

Equations (38–40) can be written in matrix form which facilitates solution, particularly if we seek the behavior at special frequencies. The equations are of the form

$$\begin{bmatrix} \alpha_{11} + \beta & 0 & \alpha_{13} - z_c\beta \\ 0 & \alpha_{22} + \gamma & \alpha_{23} + x_c\gamma \\ \alpha_{31} - z_c\beta & \alpha_{32} + x_c\gamma & \alpha_{33} + x_c^2\gamma + z_c^2\beta \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{z} \\ \bar{\theta} \end{pmatrix} = \begin{pmatrix} \bar{X}_w \\ \bar{Z}_w \\ \bar{M}_w \end{pmatrix} \tag{41}$$

where

$$\beta = \mu'\omega c\cdot\cot(\omega l/c)$$
$$\gamma = \mu\omega\kappa\cdot\cot(\omega l/\kappa)$$

and the α_{ij} can be identified by comparison with Eqs. (38–40). If we designate D as the determinant of the coefficients, we can isolate the effect of β by eliminating it from all elements but one. Multiplying the first row by z_c

and adding to the third row

$$D = \begin{vmatrix} \alpha_{11} + \beta & 0 & \alpha_{13} - z_c\beta \\ 0 & \alpha_{22} + \gamma & \alpha_{23} + x_c\gamma \\ z_c\alpha_{11} + \alpha_{31} & \alpha_{32} + x_c\gamma & z_c\alpha_{13} + \alpha_{33} + x_c^2\gamma \end{vmatrix}$$

Now multiply the first column by z_c and add to the third column to get

$$D = \begin{vmatrix} \alpha_{11} + \beta & 0 & z_c\alpha_{11} + \alpha_{13} \\ 0 & \alpha_{22} + \gamma & \alpha_{23} + x_c\gamma \\ z_c\alpha_{11} + \alpha_{31} & \alpha_{32} + x_c\gamma & z_c(z_c\alpha_{11} + \alpha_{13} + \alpha_{11}) + \alpha_{33} + x_c^2\gamma \end{vmatrix} \quad (42)$$

From Eq. (42) one can now pick out by inspection the dominant term for those frequencies at which $\omega l/c = n\pi$ ($n = 1,2,3$) or for which $\beta \to \pm\infty$ and γ is bounded assuming $\kappa \neq c$. It is the minor of the first element; hence as $\beta \to \infty$

$$D \to \{(\alpha_{22} + \gamma)(\alpha_{33} + z_c^2\alpha_{11} + x_c^2\gamma + (\alpha_{13} + \alpha_{31})z_c) - (\alpha_{23} + x_c\gamma)(\alpha_{32} + x_c\gamma)\}\beta \quad (43)$$

$D \to D_\beta \cdot \beta$ where D_β is the quantity in brackets (43a)

Analagously the behavior of D as $\gamma \to \infty$ is

$$D \to \{(\alpha_{11} + \beta)[\alpha_{33} - x_c(\alpha_{32} + \alpha_{23} - x_c\alpha_{22}) + z_c^2\beta] - (\alpha_{13} - z_c\beta)(\alpha_{31} - z_c\beta)\}\gamma \quad (44)$$

or $D \to D_\gamma \cdot \gamma$ where D_γ is the constant in brackets (44a)

As the determinant D is common to each of the system responses, we may now find the buoy surge, heave and pitch motions and investigate these as β and γ separately approach infinity.

The surge response is

$$\bar{x} = \frac{\begin{vmatrix} \bar{X}_w & 0 & \alpha_{13} - z_c\beta \\ \bar{Z}_w & \alpha_{22} + \gamma & \alpha_{23} + x_c\gamma \\ \bar{M}_w & \alpha_{32} + x_c\gamma & \alpha_{33} + x_c^2\gamma + z_c^2\beta \end{vmatrix}}{D} \quad (45)$$

Multiplying the first row by z_c and adding to the third row gives

$$\bar{x} = \frac{\begin{vmatrix} \bar{X}_w & 0 & \alpha_{13} - z_c\beta \\ \bar{Z}_w & \alpha_{22} + \gamma & \alpha_{23} + x_c\gamma \\ \bar{M}_w + z_c\bar{X}_w & \alpha_{32} + x_c\gamma & \alpha_{33} + x_c^2\gamma + z_c\alpha_{13} \end{vmatrix}}{D} \quad (46)$$

Then

$$\underset{(\beta \to \infty)}{\bar{x} \to} \frac{\begin{vmatrix} Z & \alpha_{22} + \gamma \\ M + z_c X & \alpha_{32} + x_c\gamma \end{vmatrix}(\alpha_{13} - z_c\beta) - \begin{vmatrix} \bar{X}_w & 0 \\ \bar{M}_w & \alpha_{32} + x_c\gamma \end{vmatrix}(\alpha_{23} + x_c\gamma) + \begin{vmatrix} \bar{X}_w & 0 \\ \bar{Z}_w & \alpha_{22} + \gamma \end{vmatrix}(\alpha_{33} + x_c^2\gamma + z_c\alpha_{13})}{D_\beta \cdot \beta} \quad (47)$$

and

$$\lim_{\beta \to \infty} \bar{x} = -z_c \frac{\begin{vmatrix} \bar{Z}_w & \alpha_{22} + \gamma \\ \bar{M}_w & \alpha_{32} + x_c\gamma \end{vmatrix}}{D_\beta} \quad (48)$$

In a like manner

$$\underset{(\gamma \to \infty)}{\bar{x} \to} \frac{\begin{vmatrix} \bar{X}_w & \alpha_{13} - z_c\beta \\ \bar{M}_w & \alpha_{33} + z_c^2\beta - x_c(\alpha_{23} + \alpha_{32}) + x_c^2\alpha_{22} \end{vmatrix}(\alpha_{22} + \gamma) - \begin{vmatrix} \bar{X}_w & \alpha_{13} - z_c\beta \\ \bar{Z}_w & \alpha_{23} - x_c\alpha_{22} \end{vmatrix}(\alpha_{32} - \alpha_{22})}{D_\gamma \cdot \gamma} \quad (49)$$

The heave response is

$$\bar{z} = \frac{\begin{vmatrix} \alpha_{11} + \beta & \bar{X}_w & \alpha_{13} - z_c\beta \\ 0 & \bar{Z}_w & \alpha_{23} + x_c\gamma \\ \alpha_{31} - z_c\beta & \bar{M}_w & \alpha_{33} + x_c^2\gamma + z_c^2\beta \end{vmatrix}}{D} \quad (50)$$

For $\beta \to \infty$ this can be written as

$$\underset{(\beta \to \infty)}{\bar{z} \to} \frac{\begin{vmatrix} \bar{Z}_w & \alpha_{23} + x_c\gamma \\ \bar{M}_w + z_c\bar{X}_w & \alpha_{33} + x_c^2\gamma + z_c(\alpha_{31} + \alpha_{13}) + z_c^2\alpha_{11} \end{vmatrix}(\alpha_{11} + \beta) + \begin{vmatrix} X & \alpha_{13} + z_c\alpha_{11} \\ Z & \alpha_{23} + x_c\gamma \end{vmatrix}(\alpha_{31} + z_c\alpha_{11})}{D_\beta \cdot \beta} \quad (51)$$

Again it is seen that the response is bounded for $\beta \to \infty$.

Rearrangement of the numerator to isolate the term involving γ gives

$$\underset{(\gamma \to \infty)}{\bar{z} \to} \frac{\begin{vmatrix} \alpha_{11} + \beta & \bar{X}_w \\ \alpha_{31} - z_c\beta & \bar{M}_w - x_c\bar{Z}_w \end{vmatrix}(\alpha_{23} + x_c\gamma) + \begin{vmatrix} \alpha_{11} + \beta & \alpha_{13} - z_c\beta \\ \alpha_{31} - z_c\beta & \alpha_{33} + z_c^2\beta - x_c\alpha_{23} \end{vmatrix} Z}{D_\gamma \cdot \gamma} \quad (52)$$

and this leads also to a bounded limit.

The pitch responses are

$$\underset{(\beta\to\infty)}{\theta\to}\ \frac{\begin{vmatrix}\alpha_{22}+\gamma & Z_w\\ \alpha_{32}+x_c\gamma & M_w+z_c\bar{X}_w\end{vmatrix}(\alpha_{11}+\beta)+\begin{vmatrix}0 & X\\ \alpha_{22}+\gamma & Z\end{vmatrix}(\alpha_{31}+z_c\alpha_{11})}{D_\beta\cdot\beta} \quad (53)$$

and

$$\underset{(\gamma\to\infty)}{\bar{\theta}\to}\ \frac{\begin{vmatrix}\alpha_{11}+\beta & X\\ \alpha_{31}-z_c\beta & M-x_cZ\end{vmatrix}(\alpha_{22}+\gamma)-\begin{vmatrix}\alpha_{11}+\beta & X\\ 0 & Z\end{vmatrix}(\alpha_{32}+x_c\alpha_{22})}{D_\gamma\cdot\gamma} \quad (54)$$

These are also bounded as can be seen by inspection.

Looking next to the values of the velocities, angle and tension variations at the top of the cable, we may express these as

$$u = i(\bar{z} + x_c\bar{\theta})\gamma e^{i\omega t}/\mu\kappa \quad (55)$$

$$v = -i(\bar{x} - z_c\bar{\theta})\beta e^{i\omega t}/\mu' c \quad (56)$$

$$\phi = -(\bar{x} - z_c\bar{\theta})\beta e^{i\omega t}/\mu' c^2(1+\epsilon_0) \quad (57)$$

$$T = AE(\bar{z} + x_c\bar{\theta})\gamma e^{i\omega t}/\mu\kappa^2 \quad (58)$$

These appear to become unbounded in pairs as either γ or $\beta \to \infty$, especially since we have found that $\bar{x}$, $\bar{z}$ and $\bar{\theta}$ are always bounded. The only possibility for bounded answers is if

$$\bar{z} + x_c\bar{\theta} \to \text{const}/\gamma \ \text{ as } \ \gamma \to \infty \quad (59)$$

and

$$\bar{x} - z_c\bar{\theta} \to \text{const}/\beta \ \text{ as } \ \beta \to \infty \quad (60)$$

If we multiply Eq. (54) (for $\bar{\theta}$) by x_c and add to Eq. (52), we see that indeed the terms involving γ in the numerator drop out by subtraction! Thus, the required behavior is provided and, hence, both u and T will be bounded for $\gamma \to \infty$ (and also for $\beta \to \infty$). Similarly, by multiplying Eq. (53) by z_c and subtracting from Eq. (47), we find that the terms involving β in the numerator drop out leaving only an inverse dependence on β. Hence

$$\lim_{\gamma\to\infty}(\bar{z} + x_c\bar{\theta})\gamma = \text{a constant and } \lim_{\beta\to\infty}(\bar{x} - z_c\bar{\theta})\beta = \text{a constant}$$

and u, v, ϕ, and T are all bounded.

All that can be concluded from the foregoing is that the use of the cable equations in which damping forces have been ignored *may* be useful when coupled to a buoy or a ship. When damping is included in the cable equations, these seeming infinities will not appear and, moreover, in many cases the cable force rates will be overwhelmed by those of the buoy. The foregoing will probably have relevance for cases in which the cable dominates the buoy. Criteria for dominance of one component over the other are given in a later section for the general case. Before passing to the case of a cable with arbitrary scope, there are a few remaining features of the vertical, taut cable which are intriguing and are important to observe in order to understand the behavior of numerical solutions which may be secured from computer runs.

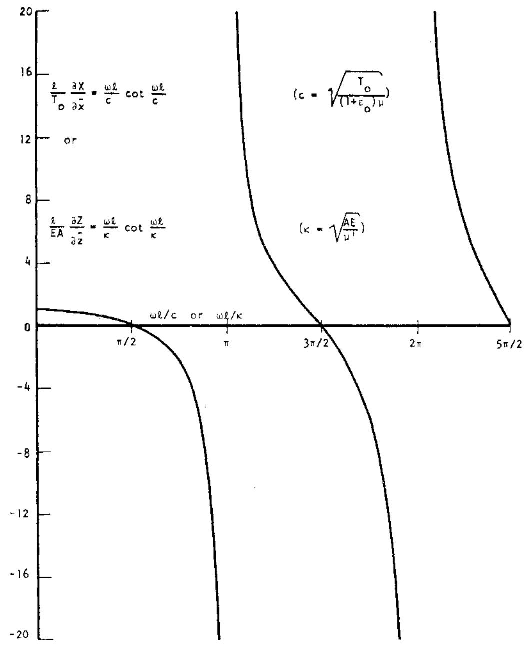

Fig. 2 Dimensionless force rates generated by a taut elastic cable as a function of $\omega l/c$ and $\omega l/\kappa$.

Frequency-Dependent Positive and Negative Springs in Certain Frequency Ranges

Returning to the coupled Eqs. (38) and (39), let us pick out the underscored, or cable-supplied terms, in surge and heave and designate these by X and Z, respectively, i.e.

$$X = \mu'\cdot\omega c\cdot\cot\frac{\omega l}{c}\cdot\bar{x} - \mu'\omega c z_c\cdot\cot\frac{\omega l}{c}\cdot\bar{\theta} \quad (61)$$

$$Z = \mu\omega\kappa\cdot\cot\frac{\omega l}{\kappa}\cdot\bar{z} + \mu\omega\kappa x_c\cot\frac{\omega l}{\kappa}\cdot\bar{\theta} \quad (62)$$

From these expressions, we can deduce the following force rates or derivatives

$$\partial X/\partial\bar{x} = \mu\cdot\omega^{c}\cdot\cot\omega l/c \ ; \ \partial X/\partial\bar{\theta} = -z_c\partial X/\partial\bar{x} \quad (63)$$

and

$$\partial Z/\partial\bar{z} = \mu\omega\kappa\cdot\cot\omega l/\kappa \ ; \ \partial Z/\partial\theta = x_c\partial Z/\partial\bar{z} \quad (64)$$

These are "spring" force rates which are not only frequency-dependent but also possess the chameleon property of changing sign in easily identifiable ranges of the parameters $\omega l/c$ and $\omega l/\kappa$. The limits for $\omega = 0$ are of first interest. These are found to be [upon use of the expressions for c and κ, Eqs. (22) and (23)]

$$\lim_{\omega\to 0}\partial X/\partial\bar{x} = T_0/l \ ; \ \lim\partial Z/\partial\bar{z} = EA/l \quad (65)$$

This suggests dividing each by the appropriate zero-frequency limit to normalize these spring rates as

$$l/T_0\cdot\partial X/\partial\bar{x} = \omega l/c\cdot\cot\omega l/c\ ;$$
$$l/EA\cdot\partial Z/\partial\bar{z} = \omega l/\kappa\cdot\cot\omega l/\kappa \quad (66)$$

The branches of these functions are given by the single plot in Fig. 2. It is clear that in the interval $0 \le \omega l/c \le \pi/2$, $0 \le \partial X/\partial\bar{x} \le T_0/l$ and for $0 \le \omega l/\kappa \le \pi/2$, $0 \le \partial Z/\partial\bar{z} \le EA/l$ and the fall-off from the static values is fairly slow. In the interval $\pi/2$ to π, each of these force rates becomes negative and approach negative infinity as the respective arguments approach π. The rates then fall through positive values to zero in the argument range from π to $3\pi/2$ and become negative again in the fourth quadrant $3\pi/2$ to 2π, and, finally, beginning the cycle of variations when passing from 2π to $5\pi/2$, etc.

Thus, the effect of transverse waves in the cable is to provide a decreasing restoring force in surge which then becomes a driving (negative spring) force as the indicated frequency-dependent arguments pass the indicated thresholds. Similarly, the effect of longitudinal or elastic waves is to change the action of the heave "spring" from a restorer to a driver. If pitch coupling in the surge and heave equations were to be absent, then one or the other mode, depending on the parameters $\omega l/c$ and $\omega l/\kappa$, can in principle be unstable in the linear sense! In the more realistic representation of the cable mechanics, when hydrodynamic damping is included in the cable equations, we may expect these effects to be modified. For sake of brevity, the inclusion of cable damping will not be dealt with here. It is appreciated that especially in the case of plastic cables the exclusion of damping is really not admissible.

One may reasonably ask-will the first threshold where these spring rates change sign be achieved in practice? The answer appears to be yes. Consider, for example, a vertical nylon line, ⅝ in. in diameter, having a length l = 5000 ft, a mass density per foot of 0.0034 (weight in air, 0.11 lb/ft.), with an elastic modulus E = 80,000 lbs/in.^{2} and an equilibrium tension of 2000 lb. Then the frequency given by

$$\omega l/\kappa = \pi/2 \tag{67}$$

is found to be

$$\omega = 0.84 \tag{68}$$

and, as the relationship between deep-water gravity, wavelength λ and frequency is

$$\lambda = 202.2/\omega^2 \tag{69}$$

we see that for this frequency

$$\lambda = 287 \text{ ft} \tag{70}$$

which is a wave length of significance. Hence, for waves shorter than this, the vertical restoring force rate of the cable will be negative. Similarly, the corresponding frequency range over which the surge force rate is negative is

$$\pi/2 < \omega l/c < 3\pi/2$$

which works out to be

$$0.16 < \omega < 0.96$$

or for wave lengths

$$219 < \lambda < 7898 \text{ ft} \tag{71}$$

Sight must not be lost of the fact that, for a nylon cable, the static restoring forces are extremely small being

$$\frac{EA}{l} = 80{,}000 \times 0.306/5000 = 0.58 \text{ lbs/ft} \tag{72}$$

and

$$\frac{T_0}{l} = 2000/5000 = 0.40 \text{ lbs/ft} \tag{73}$$

while, on the other hand, at $\omega l/c$, $\omega l/\kappa = n\pi$, (n = 1,2,3 . . .) these force rates become unbounded. Clearly, the effect of cable damping should be evaluated; it has been omitted here to keep the analysis very simple.‡

Before passing to the general case it is of interest to illustrate the rôle of the cable in the simplest case of pure heaving of a buoy in waves with an elastic cable whose equilibrium altitude is essentially vertical. Nath[2] has treated this special case in his broad review of methods for calculating cable-buoy motions and tensions. If we return to Eq. (39) and restrict the motion to pure heaving ($\bar{\theta}$ = 0) the complex heave amplitude can be solved for to give,

$$\bar{z} = \frac{\bar{Z}_w}{(A_{26} - \omega^2 A_{24} + i\omega A_{25}) + \mu\omega\kappa \cot\dfrac{\omega l}{\kappa}} \tag{73a}$$

Nath's result is recaptured if the buoy damping coefficient A_{25} is taken to be zero and if the wave exciting force Z_w is taken to be only that produced by the buoyancy force provided by the wave, Ka_0 (where K is the product of the weight density of the sea water and the water plane area of the buoy, and a_0 is the amplitude of a simple cosine wave, in Nath's notation). (It is an over simplification to take the wave exciting force as due to buoyancy alone, since the wave generates forces which are the sum of effects of the wave acceleration, wave velocity and wave displacement as has been verified from experiments with models held captive in waves.)

It is now possible to give a very straightforward interpretation of the effect of the cable on the heave response. Divide through by Z_w and the buoyancy force rate A_{26} (K in Nath's notation) and obtain (upon dropping A_{25} for simplicity)

$$\frac{\bar{z}}{(\bar{Z}_w/A_{26})} = \frac{1}{1 + (\omega\mu\kappa/A_{26})\cot(\omega l/\kappa) - (\omega^2 A_{24}/A_{26})}; \quad \kappa = (EA/\mu)^{1/2} \tag{73b}$$

The quantity $\bar{Z}/A_{26}$ is recognized as the static deflection of the buoy alone; the right side is then the frequency-dependent magnification factor; $\omega^2 A_{24}/A_{26}$ is the ratio of the "inertial force rate" to the static force rate of the buoy alone, and $(\omega\mu\kappa/A_{26})$ cot $(\omega l/\kappa)$ is the increase in the system stiffness provided by the cable. It is important to note that

$$\lim_{\omega \to 0} \left\{ \frac{\bar{z}}{\bar{Z}_w/A_{26}} \right\} = \frac{1}{1 + EA/l/A_{26}} \tag{73c}$$

where EA/l is the static stiffness of the cable, i.e., the stiffness at zero frequency. It is also significant that the cable length only enters the solution in the argument of the cotangent. Other forms of the response for this simple system given in the literature tend to befog the rôle of the various cable parameters by manipulations to form additional dimensionless ratios, thereby apparently involving the length, for example, in various terms, (see for example, Ref. 2).

‡The inclusion of damping forces along and normal to the cable elements can be readily worked out for the case of a straight cable. However the formulations used to date are quasi-static and do not take account of vortex shedding and nonlinear phenomena in an entirely rational way.

We may now turn to a brief discussion of a procedure which enables one to deal with the quite general case of a cable having an arbitrary catenary while still retaining some insight to the way in which the cable effectively modifies the buoy dynamical and static characteristics.

Procedure for the General Case

The limitation in the foregoing analysis to the case of a vertical, weightless cable, while yielding some insight to the rôle of the cable in the buoy motions, is, of course, too restrictive to be practical. If we now allow the cable to have a fairly arbitrary catenary (excluding the case where part of the cable lies on the sea floor) and otherwise arbitrary weight and elastic characteristics, then the problem of finding the buoy motions when tethered in a specified seaway must be resolved by numerical solutions of the coupled buoy-cable equations. Yet even in this case it is possible to put in evidence the alteration of the buoy matrix provided by the cable.

To make this explicit, we may note that the solution of the general cable equations (when linearized) depend only on the velocity components imposed at each end. When the lower end is anchored, then the tension and perturbed angle at the buoy depend only on the tangential and normal velocities imposed at the upper end of the cable. But these components are completely specified by the horizontal and vertical velocities and a specified equilibrium angle at the top. The forces X_m and Z_m can therefore be written as

$$X_m = (\partial X_m/\partial \dot{x})\dot{x} + (\partial X_m/\partial \dot{z})\dot{z} + \ldots \quad (74)$$

$$\overline{Z}_m = (\partial Z_m/\partial \dot{x})\dot{x} + (\partial Z_m/\partial \dot{z})\dot{z} + \ldots \quad (75)$$

where $\dot{x}$, $\dot{z}$ are the horizontal and vertical velocity components of the cable attachment point on the buoy, and the steady state term, plus the higher order terms, have been omitted.

The relationships of X_m and Z_m to the cable perturbation tension and angle are

$$\overline{X}_m = X = -(T\cos\phi_0 - T_0\phi\sin\phi_0) \quad (76)$$

$$\overline{Z}_m = Z = -(T\sin\phi_0 + T_0\phi\cos\phi_0) \quad (77)$$

T and ϕ evaluated at top of cable

and the relations connecting $\dot{x}$ and $\dot{z}$ to the buoy displacement are:

$$\dot{x} = i\omega(\overline{x} - z_c\overline{\theta})\,;\ \dot{z} = i\omega(\overline{z} + x_c\ddot{\theta})$$

(Note: In order to facilitate typing, we shall omit bars and subscripts from this point on.) Hence, the forces as a function of buoy surge, $\bar{x}$, heave, $\bar{z}$, and pitch $\bar{\theta}$ amplitudes are

$$X = i\omega\{(\frac{\partial X}{\partial \dot{x}})(\overline{x} - z_c\overline{\theta}) + (\frac{\partial X}{\partial \dot{z}})(\overline{z} + x_c\overline{\theta})\} \quad (78)$$

$$Z = i\omega\{\frac{\partial Z}{\partial \dot{x}}(\overline{x} - z_c\overline{\theta}) + \frac{\partial Z}{\partial \dot{z}}(\overline{z} + x_c\overline{\theta})\} \quad (79)$$

or

$$X = i\omega\{(\frac{\partial X}{\partial \dot{x}})\overline{x} + (\frac{\partial X}{\partial \dot{z}})\overline{z} + [x_c(\frac{\partial X}{\partial \dot{z}}) - z_c(\frac{\partial X}{\partial \dot{x}})]\overline{\theta}\} \quad (80)$$

and

$$Z = i\omega\{(\frac{\partial Z}{\partial \dot{x}})\overline{x} + (\frac{\partial X}{\partial \dot{z}})\overline{z} + [x_c(\frac{\partial Z}{\partial \dot{z}}) - z_c(\frac{\partial X}{\partial \dot{x}})]\overline{\theta}\} \quad (81)$$

Here the four force-velocity derivatives or impedances are obtained by applying unit horizontal and vertical velocities to the upper end of the cable in its equilibrium state at a number of discrete frequencies to characterize the cable over the entire frequency domain of interest. The individual derivatives each have forms like

$$\left.\frac{\partial X}{\partial \dot{x}}\right|_{\dot{z}=0} = -\frac{\partial T}{\partial \dot{x}}\cos\phi_0 - T_0\frac{\partial\phi}{\partial \dot{x}}\sin\phi_0 \quad (82)$$

Thus, the cable equations are solved for $\dot{x} = 1$ and the complex amplitudes of the tension $T(l)$ and the angle $\phi(l)$ given by the computer are the values of $\partial T/\partial \dot{x}$ and $\partial\phi/\partial \dot{x}$.

The buoy equations of motion as expressed by Eqs. (63) and (64) of Ref. 1 are

$$[A]\begin{pmatrix}\overline{x}\\ \overline{z}\\ \overline{\theta}\end{pmatrix} = \begin{pmatrix}X + \overline{X}_w\\ Z + \overline{Z}_w\\ M + \overline{M}_w\end{pmatrix} \quad (83)$$

where X, Z, M are the cable applied forces and moment X_w, Z_w, M_w are the wave forces on the buoy,§

$$[A] = \begin{vmatrix} A_{13}' & 0 & A_{17}' \\ 0 & A_{24}' & A_{27}' \\ A_{31}' & A_{34}' & A_{37}' \end{vmatrix} \quad (84)$$

and the A_{ij} are frequency-dependent coefficients defined by the buoy hydrodynamic and hydrostatic properties.

Since X, Z, and M are now expressed in terms of the unknown complex amplitudes of the buoy motion, we can transfer the terms to the left side of the motion Eq. (83) to generate an altered matrix $[A']$ which is explicitly given by

$$[A'] = \begin{vmatrix} A_{13}' - i\omega\frac{\partial X}{\partial \dot{x}} & -i\omega\frac{\partial X}{\partial \dot{z}} & A_{17}' - i\omega[x_c\frac{\partial X}{\partial \dot{z}} - z_c\frac{\partial X}{\partial \dot{x}}] \\ -i\omega\frac{\partial Z}{\partial \dot{x}} & A_{24}' - i\omega\frac{\partial X}{\partial \dot{z}} & A_{27}' - i\omega[x_c\frac{\partial X}{\partial \dot{z}} - z_c\frac{\partial Z}{\partial \dot{x}}] \\ A_{31}' - i\omega(x_c\frac{\partial Z}{\partial \dot{x}} - z_c\frac{\partial X}{\partial \dot{x}}) & A_{34}' - i\omega(x_c\frac{\partial Z}{\partial \dot{z}} - z_c\frac{\partial X}{\partial \dot{z}}) & A_{37}' - i\omega(x_c^2\frac{\partial Z}{\partial \dot{z}} + z_c^2\frac{\partial X}{\partial \dot{x}} - x_c z_c(\frac{\partial X}{\partial \dot{z}} + \frac{\partial Z}{\partial \dot{x}})) \end{vmatrix} . \quad (85)$$

§Wave forces on the cable are neglected here.

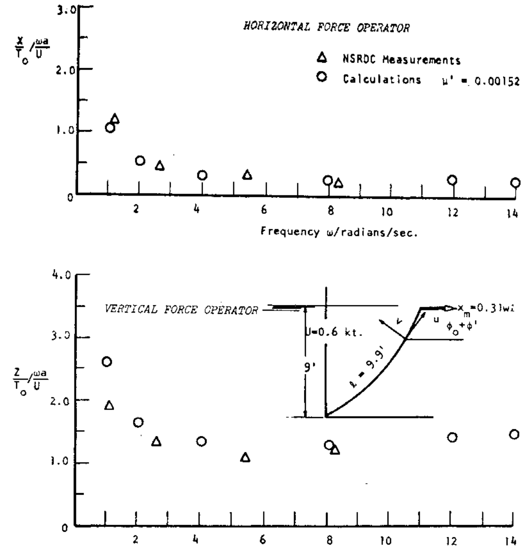

Fig. 3 Force operators generated by harmonic vertical oscillation for open link (model) chain. Current U = 0.6 kt. Scope = 1.1↑ w = 0.0362 lb/ft. C_D = 0.92 $x_m/wl = T_{ox}/wl$ = 0.34, l = 9.9 ft.

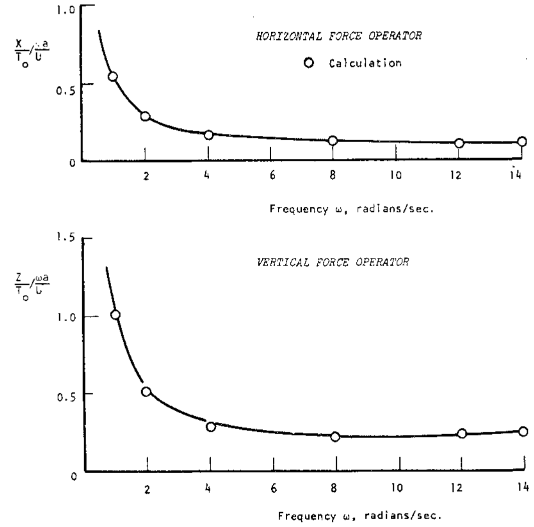

Fig. 4 Force operators generated by harmonic oscillation for open link chain. Current U = 0.6 kt. Scope 1.1 w = 0.0362 lb/ft. C_D = 0.92 $X_m/\omega l$ = 0.31 = T_{ox}/wl; l = 9.9 ft.

and the "new" buoy equations have the form

$$[A'] \cdot \begin{pmatrix} \bar{x} \\ \bar{z} \\ \bar{\theta} \end{pmatrix} = \begin{pmatrix} \bar{X}_w \\ \bar{Z}_w \\ \bar{M}_w \end{pmatrix} \quad (86)$$

so that the buoy-cable system is equivalent to a new buoy whose coefficients are altered as exhibited in Eq. (85).

We see that the effect of the cable is to provide terms in each element of the matrix because of the kinematic coupling at the buoy attachment point and also because provision of a heaving motion alone to the cable generates both a vertical and horizontal force and, similarly, a surging motion engenders both a horizontal and vertical force.¶ It is obvious that much simplification would obtain if the attachment of the buoy could be made at the center of mass of the buoy ($x_c = z_c = 0$).

Of course, many of the terms contributed by the cable may be small compared to the coefficients of the buoy. It is important to understand the effective physical alteration of the buoy provided by the cable.

Let us focus our attention on the heave equation which is now

$$\left(-i\omega \frac{\partial Z}{\partial \dot{x}}\right)\bar{x} + \left(A_{24}' - i\omega \frac{\partial Z}{\partial \dot{z}}\right)\bar{z} + \left(A_{27}' - i\omega \left(x_c \frac{\partial X}{\partial \dot{z}} - z_c \frac{\partial Z}{\partial \dot{x}}\right)\right)\bar{\theta} = \bar{Z}_w \quad (87)$$

To illuminate the discussion, let us suppose that the cross-coupling terms are weak** and that consequently we may uncouple the heave motion to yield the simple equation

$$\left\{a_{26} - a_{24}\omega^2 + i a_{25}\omega + i\omega \left(\frac{\partial T}{\partial \dot{z}} \sin\phi_0 + T_0 \cos\phi_0 \frac{\partial \phi}{\partial \dot{z}}\right)\right\}\bar{z} = \bar{Z}_w \quad (88)^{\dagger\dagger}$$

Now recalling that T and ϕ are complex amplitudes which we can express as $T = T_r + i\,T_i$; $\phi = \phi_r + i\phi_i$, then we can see that the imaginary parts of $\partial T/\partial \dot{z}$ and $\partial\phi/\partial\dot{z}$ join the spring, (a_{26}), and virtual mass terms, ($-a_{24}\omega^2$), and the real parts join the damping term, ($ia_{25}\omega$), to give

$$\left\{a_{26} - \omega\left(\frac{\partial T_i}{\partial \dot{z}} \sin\phi_0 + T_0 \cos\phi_0 \frac{\partial \phi_i}{\partial \dot{z}}\right) - a_{24}\omega^2 + i\omega\left(a_{25} + \frac{\partial T_r}{\partial \dot{z}} \sin\phi_0 + T_0 \cos\phi_0 \frac{\partial \phi_r}{\partial \dot{z}}\right)\right\}\bar{z} = \bar{Z}_w \quad (89)$$

The undamped, uncoupled natural frequencies of the buoy-cable system are given by the roots of

$$a_{24}\omega^2 + \left(\sin\phi_0 \frac{\partial T_i}{\partial \dot{z}} + T_0 \cos\phi_0 \frac{\partial \phi_i}{\partial \dot{z}}\right)\omega - a_{26} = 0 \quad (90)$$

and, since $\partial T_i/\partial z$ and $\partial\phi_i/\partial\dot{z}$ are, in general, frequency-dependent (having been found as a function of ω by exercising the cable equations for unit $\dot{z}$ input), the equation can only be solved precisely by an iterative numerical scheme.

It is clear that the alteration of the buoy-alone frequency $\omega = (\sqrt{a_{26}/a_{24}})^{1/2}$, will be significant if, and only if,

$$(\sin\phi_0 \partial T_i/\partial\dot{z} + T_0 \cos\phi_0 \partial\phi_i/\partial\dot{z})\omega \text{ is order of } a_{26}$$

It is expected that, for low frequencies, this product is nearly independent of frequency so that

$$\omega_n \sim \left(\frac{a_{22}}{a_{24}} - \frac{\left(\sin\phi_0 \frac{\partial T_i}{\partial \dot{z}} + T_0 \cos\phi_0 \frac{\partial \phi_i}{\partial \dot{z}}\right)\omega}{a_{24}}\right)^{1/2} \quad (91)$$

Thus it is important to study the behavior of $\omega\partial T_i/\partial\dot{z}$ and $\omega\partial\phi_i/\partial\dot{z}$ as a function of applied frequency. For many cases, $\phi_0 \sim \pi/2$ and the dominant term will be $\omega\partial T_i/\partial\dot{z}$ which will generally be negative (since application of upward velocity gives a downward force) and, consequently, the action of the cable will be to stiffen the system. The buoy will generally dominate the cable in heave when

$$\frac{\omega\partial T_i/\partial\dot{z}}{a_{26}} = \frac{\omega\partial T_i/\partial\dot{z}}{\rho g A_b} << 1.0 \quad (92)$$

where ρg is the weight density of water, A_b is the waterplane area of the buoy, and the buoy damping will be significantly affected only if $\partial T_r/\partial\dot{z}$ is order of a_{25}. If the buoy damping is small then the contribution of the cable to the buoy motions at resonance can be significant, but generally only in that neighborhood.

¶ For cables with arbitrary equilibrium configurations.

**That is $\partial Z/\partial x \ll \partial Z/\partial z$ and $\partial X/\partial z \ll \partial X/\partial x$ and A_{27} is also negligible.

††Here A_{24}' has been replaced by its equivalent $a_{26} - a_{24}\omega^2 + ia_{25}\omega$.

Table 1[a] Coefficients (a_{ij}) of dynamic equations for cable

	$(u)\ j = 1$	$(v)\ 2$	$(T)\ 3$	$(\phi)\ 4$
$i = 1$	0	$\frac{d\phi_0}{ds}$	$\frac{i\omega l T_0(1)}{U(1)AE}$	0
2	$-\frac{d\phi_0}{ds}$	0	0	$\frac{i\omega l}{U(1)}$
3	$\frac{i\mu\omega l U(1)}{T_0(1)}$	0	0	$\frac{wl \cos\phi_0}{T_0(1)}$
4	$\frac{lU(1)F_u}{T_0(s)}$	$\frac{i\mu'\omega l U(1) + lU(1)F_v}{T_0(s)}$	$-\frac{T_0(1)d\phi_0}{T_0(s)\cdot ds}$	$\frac{lF_\phi - w'l\sin\phi_0 + i\rho\omega UAl\cos\phi_0}{T_0(s)}$

[a] Note: Values of F_u, F_v, and F_ϕ are given in Appendix B.

Character of the Force Operators

The behavior of the horizontal and vertical forces which must be applied to the upper end of the cable to move a cable with unit harmonic *velocity* in the vertical and horizontal directions over a range of frequencies can be observed in Figs. 3 and 4. These results were obtained from numerical solutions of the cable equations which, after separation of the time function, $e^{i\omega t}$, have the form:

$$\begin{pmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \\ \frac{dT}{ds} \\ \frac{d\phi}{ds} \end{pmatrix} = [B] \begin{pmatrix} u \\ v \\ T \\ \phi \end{pmatrix} \qquad (93)$$

where the matrix B is defined by Table 1. The end conditions were alternatively specified by

For vertical excitation at unit velocity amplitude

$$\begin{aligned} u(0) &= v(0) = 0 \qquad \text{(anchor at } s/l = 0) \\ u(1) &= \sin\phi_0 \\ v(1) &= \cos\phi_0 \end{aligned} \qquad (94)$$

and for horizontal excitation at unit velocity amplitude:

$$\begin{aligned} u(0) &= v(0) = 0 \\ u(1) &= \cos\phi_0 \\ v(1) &= -\sin\phi_0 \qquad \text{(vertical velocity zero)} \end{aligned} \qquad (95)$$

The computer output gives the tension and angle variation at the upper end of the cable and, when these are resolved, one obtains the horizontal and vertical forces required at each frequency. These are then the force derivatives since the system is linear. It is seen that these rates are hyperbolic at small frequencies and tend to become constant at higher frequencies as far as the calculations were made. At extreme frequencies, the rates as graphed in Figs. 3 and 4 are expected to rise linearly because the inertia forces will rise quadratically. This behavior is not exhibited in these examples because the cable is short (10 ft) and has small mass per foot.

The successful comparison of the computed values with those obtained from measurements reported in Ref. 3 is noteworthy.‡‡

‡‡Attention is invited to the fact that calculations made for this case employed the cable weight in water throughout and did not include the term $i\rho\omega U(s)Al \text{ Cos } \varphi_0 \cdot \varphi/T_0(s)$ as derived in Appendix A, and appearing in Eq. (A34). It is expected that the effect of the corrections found there will be unimportant in the example presented in Figs. 3 and 4 because of the relatively large weight density of the chain.

It is interesting to convert Figs. 2 and 3 into forces per unit amplitude of displacement by multiplying the ordinates by the corresponding frequencies ω since each of these rates, when used in the buoy equations, are multiplied by ω. The results of this simple manipulation are given in the log-log plots of Figs. 5 and 6. The results show that the frequency dependence is generally significant only for $\omega > 2$ of for wavelengths less than 50 feet! While it may be premature to judge on the basis of such a special case of this small model cable or chain, it is the writer's speculation that, for some applications the force rates provided by the mooring line may be predominantly contributed by the static characteristics of the cable. These are inherently a part of the computer solutions since the equations do continuously converge to the static perturbations as $\omega \to 0$. This is precisely why $\omega \cdot \partial X/\partial \dot{x}$, $\omega \cdot \partial X/\partial \dot{z}$, etc., must approach a constant as $\omega \to 0$ (completely analogously with the behavior of the solution for the vertical cable as seen earlier).

It is, therefore, important to gain numerical experience by finding these cable-generated terms for a variety of cable configurations and parameters. When this is accomplished, the role of the cable in the response of a tethered buoy to waves will be much more readily understood. Moreover, it may then be possible to represent the cable by simple alterations to the matrix buoy coefficients.

Conclusions

It is concluded that the method espoused herein of placing the contributions of the cable in evidence through

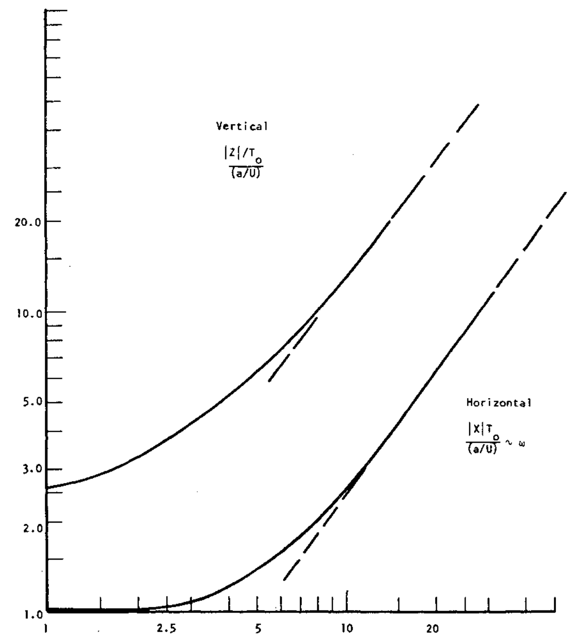

Fig. 5 Computed amplitudes of horizontal and vertical force operators per amplitude a of vertical excitation of model open link chain in a current U = 1.0 ft/sec.

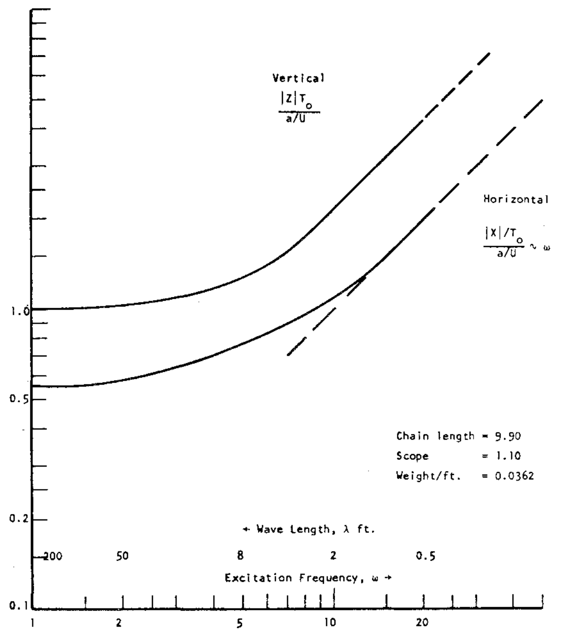

Fig. 6 Computed amplitudes of horizontal and vertical force operators per amplitude a of horizontal excitation of model open link chain in a current U = 1.0 ft/sec.

the usual expedient of a Taylor expansion of the forces, contributes materially to understanding of the nature and magnitude of the rôle played by a cable in the motions of a tethered buoy. Much more numerical experience must be obtained to isolate the importance of the various cable parameters. Further confirmation of the linear theory could be obtained from oscillation of cables in a deep river. It would seem probable that, force operators for cables may be substantially those which can be computed statically for the wavelength range of practical interest. This conjecture requires study of computer results for practical values of cable parameters to verify the range of parameters and frequencies over which it may be sufficiently valid.

Finally it is necessary to modify many of the existing computer programs to account for the presence of both w, w' in both the static and dynamic equations, as well as the influence of the static fluid pressure load at the bottom end of the cable on the distribution of static characteristics. The importance of the fluid intertia term, $i\rho\omega U(z)\, Al \cos \varphi_0(s) \cdot \varphi / T_0(s)$, must also be determined by comparative numerical evaluations.

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APPENDIX A: A New Derivation of the Equations of Motion for a Heavy, Flexible, Elastic Cable in a Dense Fluid

In all the developments of the equations for the motion of cables in water (of which the writer is aware), the fluid-dynamical effects arising from the time-dependent, relative velocities of the cable relative to the fluid have been allowed for (in the potential flow sense) by providing certain added-mass terms to the inertial part of the equations. These have been developed by appealing to certain momentum-based arguments as, for example, were advanced by Kerney[4] and followed by Goodman et al.[1]

Here, the fluid-dynamical forces arising from acceleration (excluding viscous and vortex wake effects) are accounted for by determining the integral of the pressure distribution arising from the time-dependent motion of the cable element relative to the ambient fluid motion. As a result of this (long overdue) analysis, a term which has been used previously is found not to arise, and a new term not previously incorporated is found. The effect of this term on the calculations reported in many papers (including this one!) has not been ascertained.

In addition, the fact that the static pressures are normal to the cable section has previously not generally been properly taken into account. The buoyancy force normal to the cable element is $\rho g A \cos\phi ds$.* Thus, as will be seen, the weight of the cable w (per unit length) in vacuo appears in the tangential equation whereas the weight in the fluid, w' $(= w - \rho g A)$, appears in the balance of normal forces. This fact has generally been missed heretofore, as previous investigators (including this one) have used the weight in the fluid, w', throughout. It is also believed that the static shape and distribution of static tension along a cable has not been properly calculated since the effect of static pressure on the lower end of the cable has not generally been included.

Consider an element of a flexible (no bending stiffness) cable having uniform mass μ, weight w (in vacuo), elastic modulus E (Hooke's law) acted upon by tension, weight, buoyancy and normal and tangential fluid forces, suspended in the vertical plane in a steady horizontal current of speed U which, in general, depends upon depth below the free surface of the fluid. Coordinate definitions are provided by Fig. 1. The free body diagram of the cable element is given in Fig. 7.

The basic equations of motion referred to fixed axes (x, z), as indicated in Fig. 1, are as follows:

$$\mu \frac{\partial u_x}{\partial t} = \frac{\partial T}{\partial s} \cos\phi - T \sin\phi \cdot \frac{\partial \phi}{\partial s} + F_t \cos\phi - F_n \sin\phi - \rho g A \cos\phi \sin\phi \quad \text{(A1)}$$

$$\mu \frac{\partial u_z}{\partial t} = \frac{\partial T}{\partial s} \sin\phi + T \cos\phi \cdot \frac{\partial \phi}{\partial s} + F_t \sin\phi + F_n \cos\phi - w + \rho g A \cos^2\phi \quad \text{(A2)}$$

$$\frac{\partial x}{\partial s} = (1 + \epsilon) \cos\phi \quad \text{(A3)}$$

$$\frac{\partial z}{\partial s} = (1 + \epsilon) \sin\phi \quad \text{(A4)}$$

$$\epsilon = \frac{T}{AE} \quad \text{(A5)}$$

The first two equations look after the balance of horizontal and vertical forces on any element at x, z. The next two provide the kinematic constraint between the coordinates, the stretched arc length $(1 + \epsilon)ds$ and the orienta-

*A post-galley analysis revealed that when the finite curvature is taken into account an additional term from static pressure is obtained, viz., $-\rho\gamma A(z-h)\, d\theta/ds$; see discussion at the end of this appendix.

tion $\phi(s, t)$. The last equation is simply Hooke's law for elastic deformation, ϵ being the strain. The forces F_t and F_n are the tangential and normal forces exerted on the cable as a result of its motion relative to the fluid.

It is seen that there are five unknowns, viz., u_x, u_z, T, ϕ and ϵ and there are five equations. In general, the equations are non-linear, coupled partial differential equations. Clearly, the number can be reduced to four by using Eq. (A5) in Eqs. (A3) and (A4). The problem becomes set once the conditions on the velocity components at the ends of the cable are set, namely four conditions, two on u_x and two on u_z.

Along axes tangent and normal (as shown in Fig. 1) to the cable element, we may define velocity components u and v which are related to u_x and u_z by

$$u(s,t) = u_x \cos\phi + u_z \sin\phi \quad \text{(A6)}$$

$$v(s,t) = -u_x \sin\phi + u_z \cos\phi \quad \text{(A7)}$$

If we now use these to form the sums $\dot{u} - v\dot{\phi}$ and $\dot{v} + u\dot{\phi}$ (where the dot indicates differentiation with respect to time, t), it is found that Eqs. (A1) and (A2) convert to those along the tangent and normal in the following form

$$\frac{\partial T}{\partial s} - w \sin\phi - F_t = \mu(\dot{u} - v\dot{\phi}) \quad \text{(A8)}$$

$$T\frac{\partial \phi}{\partial s} - w' \cos\phi - F_n = \mu(\dot{v} + u\dot{\phi}) \quad \text{(A9)}$$

where $w' = w - \rho g A$, the weight of the cable per unit length in the fluid.§§

Use of the relationships, $u_x = \partial x/\partial t = \dot{x}$, $u_z = \dot{z}$, allows one to manipulate Eqs. (A3) and (A4) into

$$\frac{\partial u}{\partial s} = \frac{v\partial\phi}{\partial s} + \frac{\dot{T}}{AE} \quad \text{(A10)}$$

$$\frac{\partial v}{\partial s} = \frac{u\partial\phi}{\partial s} - (1 + \frac{T}{AE})\dot{\phi} \quad \text{(A11)}$$

and Eqs. (A8) and (A9)

$$\frac{\partial T}{\partial s} = \mu(\dot{u} - v\dot{\phi}) + w \sin\phi + F_t \quad \text{(A12)}$$

$$T\frac{\partial\phi}{\partial s} = \mu(\dot{v} + u\dot{\phi}) + w' \cos\phi + F_n \quad \text{(A13)}$$

These are the forms in which the equations are generally expressed as coupled partial differential equations.

The remainder of the development is limited to that part of the normal force, F_n, which arises from the potential flow effects inasmuch as this aspect of the cable motion equations has not been considered from a purely hydromechanical viewpoint.

The pressure p generated at any point in the fluid by the motion of the cable can be calculated with respect to axes moving with an element of the cable by use of the relation

$$p(x,y,z,t) = -\rho\frac{\partial\psi}{\partial t} + \frac{1}{2}\rho(V^2 - q_r^2) + \text{constant} \quad \text{(A14)}$$

where q_r = is the magnitude of the fluid velocity at any point P relative to the moving axes; V = is the velocity of the same point regarded as fixed to the moving axes; and ψ = is the velocity potential of the cable with respect to fixed axes.

§§In previous derivations, the weight of the cable in the fluid has been used throughout.

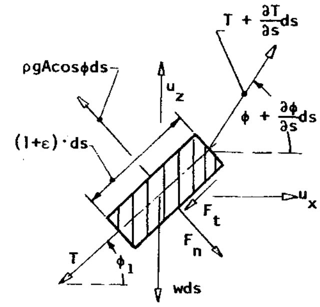

Fig. 7 Free body diagram of stressed cable element.

It is known that, in an inviscid fluid, the quadratic terms of Eq: (A14) (including the convective cross products inherent in them) do not contribute to the force when circulation is absent. Hence, the only term which can contribute to the sectional force (in such a fluid) is $-\rho\partial\psi/\partial t$ when integrated about any section of interest.

A three-dimensional velocity potential for a cable with a circular cross section can be expressed in terms of a distribution of doublets along the cable centerline with axes directed normal to the cable elements in the direction of motion. If $\sigma(s, t)$ represents the strength-density of these doublets, the velocity potential function is expressible by

$$\psi(x,y,z,t) = -\frac{1}{4\pi}\int_C \sigma(s,t)(-\sin\phi\cdot\frac{\partial}{\partial x'} + \cos\phi\cdot\frac{\partial}{\partial z'})\{(x - x')^2 + y^2 + (z - z')^2\}^{-1/2} \quad \text{(A15)}$$

where the integration is carried out along the curve of the cable C, holding time fixed. After differentiation, the dummy coordinates x', z' take on those of C.

It is now necessary to determine a relation between the velocity along the normal to any surface element of the cable and the doublet strength. In general, this would lead to an integral equation since, in principle, all doublets along the cable contribute to the normal velocity at any selected point. However, a rational simplification can be made by exploiting the fact that, almost everywhere, the radius of curvature of C is many times the diameter of the cable. Hence, the fluid motion "seen" by the point P (x, y, z) as it is moved close to the cable is that generated by a two-dimensional cylinder which appears to extend to infinity in either direction along the tangent to the cable element. Thus Eq. (A15) must contain this two-dimensional, near-field behavior.

If one takes a set of cordinates ξ and η having axes at the points $x = x'$, $y = 0$ and $z = z'$ such that ξ runs along the tangent and η along the normal to the cable, then the following relations hold:

$$\begin{aligned} \xi \cos\phi - \eta \sin\phi &= x - x' \\ \xi \sin\phi + \eta \cos\phi &= z - z' \end{aligned} \quad \text{(A16)}$$

Consider x, z, and η as fixed; then the differential arc length becomes

$$ds - d\xi$$

(as the cable appears as a straight line when x, y, z are brought close to the cable).

Upon carrying out the indicated differentiations in (A15), the limiting form of the potential is

$$\psi \to \psi_2 = \frac{-\eta\sigma(x',z',t)}{4\pi}\int_{\xi=-\infty}^{\infty}\frac{d\xi}{(\xi^2 + \eta^2 + y^2)^{3/2}} \quad \text{(A17)}$$

and the integral reduces to give

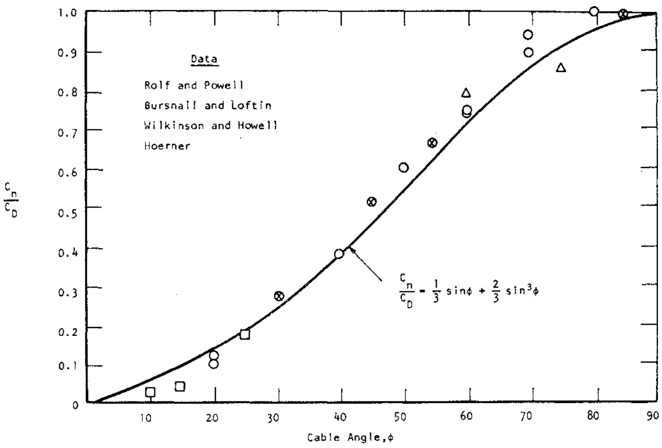

Fig. 8 Hydrodynamic normal force density at any angle ∂ in fraction of value at $\phi = \pi/2$ as interpolated by indicated formula.

$$\psi_2 = \frac{-\eta\sigma(x',z',t)}{2\pi(\eta^2 + y^2)} \tag{A18}$$

This formula could, of course, have been written down from the outset on purely physical grounds since it is the potential of a two-dimensional doublet. The flow pattern at any x', z' is two-dimensional in the η, y plane, i.e., normal to each section.

As the cable section has instantaneous normal velocity $v(s, t)$ in a steady current whose normal component is $-U \sin\phi\ (t)$ (note that the normal component depends upon time), then the velocity of the cable element relative to the fluid is

$$v(s,t) - [(-U(z)\cdot \sin\phi(s,t))]$$

The potential ψ_2 can be expressed in polar coordinates, r,θ [$r = (\eta^2 + y^2)^{1/2}$; $\theta = \tan^{-1}(y/\eta)$]. Then

$$\psi_2 = -\frac{\sigma}{2\pi}\frac{\cos\theta}{r} \tag{A19}$$

and σ must be found from the condition that

$$\left.\frac{\partial\psi_2}{\partial r}\right|_{r=a} = \frac{\sigma\cos\theta}{2\pi a^2} = (v + U\sin\phi)\cos\theta;\ (a \text{ is cable radius}) \tag{A20}$$

or

$$\sigma(s,t) = 2A(v + U\sin\phi)$$

a well-known result. However, in this case, the circular sections "think" they are moving in an onset flow which pulsates with time since ϕ varies with time.

The contribution to F_n from this unsteady potential flow is then

$$F_n = \rho a \int_0^{2\pi} \left.\dot{\psi}_2\right|_{r=a} \cos\theta d\theta$$

Using Eqs. (A19) and (A20), it is found that

$$F_n = -\rho A(\dot{v} + U\cos\phi \cdot \dot{\phi}) \tag{A21}$$

The term $-\rho A\dot{v}$ is the familiar added mass effect since ρA is the mass of the fluid displaced by the cable (per unit length).

The equations of motion now read

$$\frac{\partial T}{\partial s} = \mu(\dot{u} - v\dot{\phi}) + w\sin\phi + F_t \tag{A22}$$

$$T\frac{\partial\phi}{\partial s} = \mu'\dot{v} + (\mu u + \rho AU\cos\phi)\dot{\phi} + w\cos\phi + F_n' \tag{A23}$$

¶¶ In Refs. (1) and (4), the term $\mu v\dot{\phi}$ is given as $\mu' v\dot{\phi}$ and the term $\rho AU\cos\phi\cdot\dot{\phi}$ is not included. The error in the tangential equation does not survive linearization, but the new term $\rho AU\cos\phi\cdot\dot{\phi}$ contributes except for $\phi_0 = \pi/2$.

where μ' is the virtual mass, $\mu + \rho A$, for motion normal to the element; F_n' is the normal force from viscous flow effects.

When the motion applied to the end of the cable is a simple harmonic of small amplitudes, we may introduce

$$\begin{aligned} T &= T_0(s) + T'(s)e^{i\omega t};\ |T'(s)| << T_0(s) \\ \phi &= \phi_0(s) + \phi'(s)e^{i\omega t};\ |\phi'(s)| << \phi_0(s) \\ u &= u'e^{i\omega t} \\ v &= v'e^{i\omega t} \end{aligned} \tag{A24}$$

where the primed quantities are complex amplitudes (and only real parts are to be retained). Upon substitution of Eq. (A24) into Eqs. (A10), (A11), (A22) and (A23) and, neglecting products of these amplitudes, we can obtain the following form of the linearized equations

$$\frac{du'}{ds} = \left(\frac{d\phi_0}{ds}\right)\cdot v' + \frac{i\omega T'}{AE} \tag{A25}$$

$$\frac{dv'}{ds} = \left(\frac{d\phi_0}{ds}\right)\cdot u' + i\omega\phi' \tag{A26}$$

$$\frac{dT'}{ds} = \left(i\omega\mu + \frac{\partial F_t}{\partial u'}\right)u' + \frac{\partial F_t}{\partial v'}v' + \left(w\cos\phi_0 + \frac{\partial F_t}{\partial\phi'}\right)\phi' \tag{A27}$$

$$T_0(s)\frac{d\phi'}{ds} = \frac{\partial F_n'}{\partial u'}u' + \left(i\omega\mu' + \frac{\partial F_n'}{\partial v'}\right)v' - \left(\frac{d\phi_0}{ds}\right)T' + \left(i\rho\omega AU\cos\phi_0 + \frac{\partial F_n'}{\partial\phi'} - w'\sin\phi_0\right)\phi' \tag{A28}$$

The term T_0/AE, the static strain in Eq. (A11), has been neglected under the assumption that $T_0/AE \ll 1$ and use has been made of the static or equilibrium equations

$$\frac{dT_0}{ds} = w\sin\phi_0 + F_{t_0}(s,\phi_0) \tag{A29}$$

$$T_0\frac{d\phi_0}{ds} = w'\cos\phi_0 + F_{n_0}(s,\phi_0) \tag{A30}$$

(These equations must be solved together with specified end conditions and applied forces to yield $T_0\ (s)$ and $\phi_0(s)$ as these quantities and $d\phi_0/ds$ are needed to solve for the perturbed motions, etc.)

In the work done by the writer, the tangential drag force, F_t, has been neglected. The evaluation of the normal force derivatives $\partial F_n'/\partial u'$, $\partial F_n'/\partial v'$, $\partial F_n'/\partial\phi'$ is given in Appendix B where F_n' is designated simply by F and these derivatives are indicated by F_u, F_v and F_ϕ. Finally, if velocities are normalized by the current velocity at the surface $U(1)$, the arc length by the cable length l and, upon dropping the primes on the complex amplitude functions, we have the equations in the final form:

$$\frac{du}{ds} = 0\cdot u + \frac{d\phi_0 v}{ds} + \frac{i\omega lT}{AEU(1)} + 0\cdot\phi \tag{A31}$$

$$\frac{dv}{ds} = \frac{-d\phi_0 u}{ds} + 0\cdot v + 0\cdot T + \frac{i\omega l\phi}{U(1)} \tag{A32}$$

$$\frac{dT}{ds} = \frac{i\mu\omega lU(1)u}{T_0(1)} + 0\cdot v + 0\cdot T + \frac{\omega l\cos\phi_0\phi}{T_0(1)} \tag{A33}$$

$$\frac{d\phi}{ds} = \frac{lU(1)F_u}{T_0(s)}\cdot u + \frac{lU(1)}{T_0(s)}(i\mu'\omega + F_v)v - \frac{T_0(1)T}{T_0(s)} + \frac{l(F_\phi - w'\sin\phi_0 + i\rho\omega U(s)A\cos\phi_0)\phi}{T_0(s)} \tag{A34}$$

Table 2 Coefficients of dynamical equations for cables including terms arising from finite curvature contribution to the normal force by fluid static pressure

	u	v	T_a'	φ
$\frac{du}{ds}$	$\underline{\frac{i\omega\rho gAl\sin\varphi_0}{AE}}$	$\frac{d\varphi_0}{ds} + \underline{\frac{i\omega l\rho gA\cos\varphi_0}{AE}}$	$\frac{i\omega lT_{a0}(1)}{V(1)AE}$	0
$\frac{dv}{ds}$	$-\frac{d\varphi_0}{ds}$	0	0	$\frac{i\omega l}{V(1)}\left(1 + \underline{\frac{T_{a0} + \rho gA(z_0 - h)}{AE}}\right)$
$\frac{dT_a'}{ds}$	$\frac{lV(1)}{T_{a0}(1)}\left(i\omega\mu + \frac{\partial F_t'}{\partial\eta}\right)$	0	0	$\frac{(lw'\cos\varphi_0 + l\partial F_t'/\partial\varphi)}{T_{a0}(1)}$
$T_{a0}\frac{d\varphi}{ds}$	0	$\frac{i\omega lV(1)\mu' + lV(1)F_v}{T_{a0}(1)}$	$-\frac{d\varphi_0}{ds}$	$\frac{(i\rho A\omega lV(s)\cos\varphi_0 + lF_\varphi/\partial\varphi - w'l\sin\varphi_0)}{T_{a0}(1)}$

Notes: 1) $T_a = T - \rho gA(z - h)$, $T_{a0} = T_0 - \rho gA(z_0 - h)$, $T_a' = T' - \rho gAz'$ where T is the tension in the cable. 2) The underscored terms are those arising from the effect of cable curvature. 3) Account has been taken of the fact that $F_u = 0$ and $\partial F_t'/\partial v = 0$ where F is the normal force from relative velocity and F_t' is the tangential force from relative velocity.

The coefficients of these equations are listed in Table 1. The computer results obtained by the writer (and others) for the general case have used w' for w in Eq. (A33) and the term $i\rho\omega AU(s)\cos\phi_0$ has been omitted. For steel cables or chain, as evaluated herein, this last term is probably unimportant. For cables having small w', it may be important.

Equation (A-13) does not represent the situation when the radius of curvature is of the order of z-h or when the tension T is of the order of ρyA(h-z). A more detailed analysis (not given herein) reveals that the effect of finite radius of curvature of any cable element requires that T in Eq. A13 (and elsewhere) be replaced by $T - \rho yA\ (z\text{-}h) = T_a$, say, which might be called the apparent tension. The use of this substitution in (A12) which yields (dT/ds) = (dTa/ds) + $\rho g A\sin\varphi$ gives rise to the appearance of W' in the tangential equation. In this respect the modified equations are similar in form to those of Ref. 1. However, the presence of $z(s,t)$ gives rise to static and dynamic couplings which have not (to the writer's knowledge) been recognized previously. The implications of cable curvature are summarized in Table 2 for the dynamical equations. Discussions in connection with cable statics in this regard with L. Wagner-Smitt, Danish Ship Research Laboratory, are gratefully acknowledged.

APPENDIX B: Derivatives of Cable Normal Force Derivatives

The force normal to a cable arising from the action of current and cable motions is assumed to be

$$\frac{F}{\frac{1}{2}\rho C_D d} = (a\sin\psi + b\sin^3\psi)V^2 \qquad (B1)$$

(It has been found that this formula fits steady state normal force data for $a = 1/3$, $b = 2/3$ as seen in Fig. 8.) where ψ is the angle between the tangent to the cable and the "resultant" velocity V (specified below) at any instant, and C_0 is the section drag coefficient at $\psi = \pi/2$.

This angle is given by

$$\psi = \phi_0 + \phi - \tan^{-1}\left\{\frac{-u_z}{U - u_x}\right\} \qquad (B2)$$

where $V^2 = (U - u_x)^2 + u_z^2 \sim U^2 - 2Uu_x$

$$u_x = -v\sin(\phi_0 + \phi)$$

and (B3)

$$u_z = +v\cos(\phi_0 + \phi)$$

The cable tangential velocity u is omitted since tangential motion can produce no significant change in normal force.*

If we now simplify ψ and V^2 to

$$\psi \sim \phi_0 + \phi + \frac{v\cos\phi_0}{U} \qquad (B4)$$

$$V^2 \sim U^2 - 2U(u\cos\phi_0 - v\sin\phi_0) \qquad (B5)$$

Then writing F_u for $\partial F/\partial u$, we have

$$F_u = 0 \qquad (B6)$$

Similarly

$$\frac{F_v}{\frac{1}{2}\rho dC_D} = (a + 3b\sin^2\phi_0)\cos\phi_0\frac{\partial\psi}{\partial v}U^2 + (a + b\sin^2\phi_0)\sin\phi_0\frac{\partial}{\partial v}(V^2)$$

From Eqs. (B4) and (B5), using $a = 1/3$, $b = 2/3$

$$\frac{\partial\psi}{\partial v} = \frac{\cos\phi_0}{U}$$

$$\frac{\partial V^2}{\partial v} = 2U\sin\phi_0$$

Finally,

$$F_v = \frac{\rho dC_D}{6}(1 + 7\sin^2\phi_0 - 2\sin^4\phi_0)U \qquad (B7)$$

which gives

$$F_v(0) = \frac{\rho dC_D U}{6}$$

$$F_v\left(\frac{\pi}{2}\right) = \rho dC_D U$$

In like fashion

$$\frac{F_\phi}{\frac{1}{2}\rho dC_D} = (a + 3b\sin^2\phi_0)\cos\phi_0\frac{\partial\psi}{\partial\phi}U^2 + (a + b\sin^2\phi_0)\sin\phi_0\frac{\partial V^2}{\partial\phi}$$

with $\partial\psi/\partial\phi = 1$ and $\partial V^2/\partial\phi = 0$, we find

$$F_\phi = \frac{1}{6}\rho dC_D(1 + 6\sin^2\phi_0)\cos\phi_0 U^2 \qquad (B8)$$

*This requirement was pointed out by J. Wagner-Smith at DSRL, Denmark.